Results in Homotopy Theory of Gauge Groups

Written by Mitsunobu Tsutaya.
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I don't think that I could collect all the related works here and I think that this list might include some incorrect descriptions. Indeed, I have read only a few papers in this list. If there are any mistake here, please let me know.

1. Some selected results in chronological order

- James, 1963: A study of the space of bundle maps.
- Gottlieb, 1972: $B\mathcal{G} \simeq \mathrm{Map}(B, BG)_f$.
- Kono, 1991: The first result on the classification of the homotopy types of gauge groups.
- Crabb–Sutherland, 2000: Finiteness of homotopy equivalence types of gauge groups.

2. Homotopy types of $\mathcal{G}$

General results on homotopy types.

- $\mathcal{G}(P) \simeq \Omega\,\mathrm{Map}(B, BG)_f$ where $f\colon B \to BG$ is the classifying map of P (Gottlieb, 1972; Atiyah–Bott, 1982).
- Finiteness of homotopy types of gauge groups for G a connected Lie group and B a finite complex (Crabb–Sutherland, 2000).
- Fibrewise decomposition of the adjoint bundle induced from an automorphism on G (Kishimoto–Kono, 2010).

General homotopy decompositions over spheres.

- Theriault, 2010: p-local decomposition $\mathcal{G}$ for any simple Lie group G over S^4 such that $p > n_\ell + 1$ (p-locally trivial case) and some decompositions for p-locally non-trivial cases when $G = \mathrm{SU}(n)$.
- Kishimoto–Kono–Tsutaya, 2013: p-local decomposition $\mathcal{G} \simeq \mathcal{B}_1 \times \cdots \times \mathcal{B}_{p-1}$ for any Lie group G over S^n such that $\pi_{n-1}(G) \otimes \mathbb{Q} \cong \mathbb{Q}$.
- Kishimoto–Kono–Theriault, 2014: p-local refined decompositions of $\mathcal{G}$ for $G = \mathrm{SU}(n), \mathrm{Sp}(n)$ of any rank over S^4.

Consider the cases when $\pi_{2n-1}(G) \cong \mathbb{Z}$. Let P_k be the principal G-bundle classified by $k\epsilon\colon S^{2n} \to BG$ for a fixed generator $\epsilon \in \pi_{2n}(BG) \cong \mathbb{Z}$.

Complete classifications over S^4.

- SU(2)-bundles over S^4.
  - Kono, 1991: $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(12, k) = (12, k')$.
- SU(3)-bundles over S^4.
  - Hamanaka–Kono, 2006: $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(24, k) = (24, k')$.
- SO(3)-bundles over S^4 ($\mathrm{SO}(3) = \mathrm{PU}(2)$).
  - Kamiyama–Kishimoto–Kono–Tsukuda, 2007: $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(12, k) = (12, k')$.
- PU(3)-bundles over S^4.
  - Hasui–Kishimoto–Kono–Sato, 2016: $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(24, k) = (24, k')$.
- U(2)-bundles over S^4.
  - Cutler, 2018: $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(12, k) = (12, k')$.
- U(3)-bundles over S^4.

– Cutler, 2018: $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(24, k) = (24, k')$.

Complete classifications over spheres of other dimensions.

- SU(3)-bundles over S^6:
  - Hamanaka–Kaji–Kono, 2007: $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(120, k) = (120, k')$.
- Sp(2)-bundles over S^8.
  - Hamanaka–Kaji–Kono, 2008: $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(140, k) = (140, k')$.

Complete classifications over other manifolds.

- SU(2)-bundles over a sphere.
  - Claudio–Spreafico, 2009: Classification of the homotopy types of $\mathcal{G}$ for $B = S^n$ such that $5 < n \leq 25$ except $n = 21$.
- SU(2)-bundles over a simply connected closed 4-manifold B:
  - Kono–Tsukuda, 1996: $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(12, d(B)k) = (12, d(B)k')$, where $d(B)$ is the parity of the intersection form of B.
- SU(3)-bundles over a simply connected closed spin 4-manifold B:
  - Theriault, 2012: $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(24, k) = (24, k')$.

Complete local classifications.

- SU(5)-bundles over S^4: $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_{k'})_{(p)}$ for all prime p if and only if $(120, k) = (120, k')$.
  - Hamanaka–Kono, 2006: The order of $[\Sigma^4\mathbb{C}P^2, \mathcal{G}(P_k)]$ is determined by $(120, k)$.
  - Theriault, 2013: The order of $\langle \epsilon, \mathrm{id}\rangle$ is 120.
- Sp(2)-bundles over S^4:
  - Sutherland, 1992: If $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_{k'})_{(p)}$ for all prime p, then $(10, k) = (10, k')$.
  - Choi–Hirato–Mimura, 2008: $40\langle \epsilon, \mathrm{id}\rangle = 0$ (the precise order was not determined).
  - Theriault, 2010: $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_{k'})_{(p)}$ for all prime p if and only if $(40, k) = (40, k')$ (the order of $\langle \epsilon, \mathrm{id}\rangle$ is 40 and some observation on the homotopy sets).
- PSp(2)-bundles over S^4.
  - Hasui–Kishimoto–Kono–Sato, 2016: $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ for all prime p if and only if $(40, k) = (40, k')$.
- U(5)-bundles over S^4:
  - Cutler, 2018: $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_{k'})_{(p)}$ for all prime p if and only if $(120, k) = (120, k')$.
- SU(3)-bundles over a simply connected closed non-spin 4-manifold B:
  - Theriault, 2012: $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ for all prime p if and only if $(12, k) = (12, k')$.
- U(n)-bundles over an orientable surface:
  - Sutherland, 1992: $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ for all prime p if and only if $(n, k) = (n, k')$.
  - Theriault, 2011: Homotopy decomposition of $\mathcal{G}(P)$ for all prime p.

Rational homotopy types.

- Crabb–Sutherland, 2000: The rationalized gauge group is equivalent to that of the trivial bundle when G is a connected Lie group.
- Wockel, 2007: Rational homotopy groups of gauge groups for G a Lie group and B a sphere or a surface.
- Félix–Oprea, 2009: Rational homotopy groups of gauge groups for G a Lie group.

Other results on the homotopy types of gauge groups.

- SU(n)-bundles over S^4.

- Sutherland, 1992: The order of $\pi_{2n-4}(\mathcal{G}(P_k))$ and $\pi_{2n-2}(\mathcal{G}(P_k))$ are computed for $n > 2$. These results imply that if $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_{k'})_{(p)}$ for all prime p, then $(n(n^2-1)/(n+1,2), k) = (n(n^2-1)/(n+1,2), k')$.
- Hamanaka–Kono, 2006: The order of $[\Sigma^{2n-6}\mathbb{C}P^2, \mathcal{G}(P_k)]$ is $\frac{1}{2}(n-2)!(n(n^2-1),k)(n-1)!(n,k)$ (the orders of such groups for k and k' coincide if and only if $(n(n^2-1),k) = (n(n^2-1),k')$).
- Kishimoto–Kono–Tsutaya, 2013: A partial converse of the above results for $G = \mathrm{SU}(n)$ of "low rank".

- $\mathrm{SU}(n)$-bundles over spheres.
  - Kishimoto–Kono–Tsutaya, 2014: Some lassification of p-local homotopy types of $\mathcal{G}$ for $G = \mathrm{SU}(n)$ of "low rank" and $B = S^{2d}$ where $d = 2, 3, \ldots, n$.
  - Theriault, 2017: Some classification of p-local homotopy types of $\mathcal{G}$ for $G = \mathrm{SU}(n)$ of "low rank" and $B = S^4$. This also bounds the order of the Samelson product $S^3 \wedge \Sigma\mathbb{C}P^{n-1} \to \mathrm{SU}(n)$.
- $\mathrm{SU}(n)$-bundles over B a simply connected closed spin 4-manifold.
  - Sutherland, 1992: The order of $\pi_{2n-4}(\mathcal{G}(P_k))$ and $\pi_{2n-2}(\mathcal{G}(P_k))$ are computed for $n > 2$. These results imply that if $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_{k'})_{(p)}$ for all prime p, then $(n(n^2-1)/(n+1,2), k) = (n(n^2-1)/(n+1,2), k')$.
- $\mathrm{Sp}(n)$-bundles over B a simply connected closed spin 4-manifold.
  - Sutherland, 1992: If $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_{k'})_{(p)}$ for all prime p, then $(n(2n+1),k) = (n(2n+1),k')$ when n is even and $(4n(2n+1),k) = (4n(2n+1),k')$ when n is odd.
- $\mathrm{SU}(3)$-bundles over general spaces.
  - Kono–Theriault, 2013: The order of $\langle \mathrm{id}, \mathrm{id}\rangle$ is 120.
- $\mathrm{Sp}(3)$-bundles over S^4.
  - Cutler, 2018: $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_{k'})_{(p)}$ for all prime p if and only if $(N,k) = (N,k')$ where $N = 84, 168$ or 336.
- G_2-bundles over S^4.
  - Kishimoto–Theriault–Tsutaya, 2017: $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_{k'})_{(p)}$ for all prime p if and only if $(N,k) = (N,k')$ where $N = 84$ or 168.
- Gauge groups over non-orientable surfaces.
  - Theriault, 2013: Some homotopy decomposition and applications to the moduli space of flat connections.
- $\mathrm{U}(n)$-bundles over $\mathbb{C}P^2$.
  - Cutler, 2018: Some classification of the homotopy types and the decompositions of $\mathcal{G}$.
- p-local fibrewise triviality of adjoint bundles.
  - Kono–Tsukuda, 2010: Under some assumptions for the base spaces. Explicit applications to $\mathrm{SU}(2)$-bundles over 4-manifolds and to universal bundles.

3. Multiplicative properties of $\mathcal{G}$

General results.

- Finiteness of H-types of gauge groups for G a Lie group and B a finite complex (Crabb–Sutherland, 2000).
- Finiteness of A_n-types ($n < \infty$) of gauge groups for G a Lie group and B a finite complex (Tsutaya, 2012).
- Relation between the splitting as A_n-spaces and the homotopy commutativity (Kishimoto–Kono, 2010).

Classification of A_n-types.

- Crabb–Sutherland 2000: H-types of $\mathcal{G}(P)$ for $B = S^4$, $G = \mathrm{SU}(2)$.
- Tsutaya, 2012, 2012, 2015, 2018:
  - Some classifications of p-local A_n-types of $\mathcal{G}(P)$ for $B = S^4$, $G = \mathrm{SU}(2)$.
  - Complete classification of fibrewise A_3-types of the adjoint bundles for $B = S^4$, $G = \mathrm{SU}(2)$.

Homotopy commutativities.

- Crabb–Sutherland, 1992: Homotopy commutativity of SU(2)-bundles over S^4 and some other cases.
- Crabb–Sutherland–Zhang, 1999: Homotopy nilpotency of $\mathcal{G}$.
- Kishimoto–Kono–Theriault 2013: p-local homotopy commutativity of $\mathcal{G}$ over S^4.
- Hasui–Kishimoto–Tsutaya 2019: p-local higher homotopy commutativity of $\mathcal{G}$ for $P = EG|_{B_nG}$ and bundles over spheres.

4. (Co)homology of $\mathcal{G}$

- Terzić, 2005: rational cohomology of $\mathcal{G}(P)$ for B a compact simply connected 4-manifold, G a semisimple compact simply connected Lie group.
- Choi, 2008: mod p homology (Pontryagin ring) of $\mathcal{G}(P)$ for $B = S^4$, $G = \mathrm{SU}(n)$.
- Choi, 2008: mod p homology (Pontryagin ring) of $\mathcal{G}(P)$ for $B = S^4$, $G = G_2$.
- Theriault, 2012, mod p (odd prime) homology (Pontryagin ring) of $\mathcal{G}(P)$ for B a simply connected closed 4-manifold, $G = \mathrm{SU}(n)$, $\mathrm{Sp}(n)$ and $\mathrm{Spin}(n)$ except for some cases.

5. Homotopy types of $B\mathcal{G} \simeq \mathrm{Map}(B, BG)_f$

- Gottlieb, 1972; Atiyah–Bott, 1982: $B\mathcal{G}(P) \simeq \mathrm{Map}(B, BG)_f$ where f is the classifying map of P.
- On $\mathrm{Map}(S^4, B\,\mathrm{SU}(2))$.
  - Tsukuda, 1996: If $\mathrm{Map}(S^4, B\,\mathrm{SU}(2))_k \simeq \mathrm{Map}(S^4, B\,\mathrm{SU}(2))_{k'}$, then $(p, k) = (p, k')$ for any prime p.
  - Tsukuda, 2001: $\mathrm{Map}(S^4, B\,\mathrm{SU}(2))_k \simeq \mathrm{Map}(S^4, B\,\mathrm{SU}(2))_{k'}$ if and only if $|k| = |k'|$.
  - Tsutaya, 2012: Correction of 2-local computation in Tsukuda's result.
- $\mathrm{Map}(B, B\,\mathrm{SU}(2))$ for B a simply connected closed 4-dimensional manifold.
  - Tsukuda, 1996: $\mathrm{Map}(S^4, B\,\mathrm{SU}(2))_k \simeq \mathrm{Map}(S^4, B\,\mathrm{SU}(2))_{k'}$ implies $(k, p) = (k', p)$ for any prime p (Observations on the k-invariants).
  - Kono–Tsukuda, 2000:
    * $\mathrm{Map}(B, B\,\mathrm{SU}(2))_k \simeq \mathrm{Map}(B, B\,\mathrm{SU}(2))_{k'}$ if and only if $|k| = |k'|$ when B admits an orientation-reversing homotopy equivalence.
    * $\mathrm{Map}(B, B\,\mathrm{SU}(2))_k \simeq \mathrm{Map}(B, B\,\mathrm{SU}(2))_{k'}$ if and only if $k = k'$ when B does not admit any orientation-reversing homotopy equivalence.
- $\mathrm{Map}(B, B\,\mathrm{U}(n))$ for B a closed orientable surface.
  - Sutherland, 1992: $\mathrm{Map}(B, B\,\mathrm{U}(n))_k \simeq \mathrm{Map}(B, B\,\mathrm{U}(n))_{k+n}$ for any $k \in \mathbb{Z}$. If $(n, k) = (n, k')$, then $(\mathrm{Map}(B, B\,\mathrm{U}(n))_k)^\wedge_p \simeq (\mathrm{Map}(B, B\,\mathrm{U}(n))_{k'})^\wedge_p$ for any prime p.
- Kishimoto–Tsutaya, 2016: There are infinitely many different homotopy types among the path components of $\mathrm{Map}(S^n, BG)$ if it has infinitely many path components.

Stable homotopy types.

- Bauer–Crabb–Spreafico, 2001: Stable homotopy decompositions of $B\mathcal{G}$ for $G = \mathrm{U}(2), \mathrm{SO}(3)$ and $B = S^2$.

6. (Co)homology of $B\mathcal{G} \simeq \mathrm{Map}(B, BG)_f$

Over 4-dimensional manifolds.

- Masbaum, 1991: Some computations on $H^*(\mathrm{Map}(B, B\,\mathrm{SU}(2)))$ is done for various coefficients and B a 4-dimensional manifold.
- Tsukuda, 1997: On the cohomology of $B\mathcal{G}$ for $G = \mathrm{SO}(3)$ and $B = S^2$.
- Choi, 1997: Mod 2 and rational homology groups of $\mathrm{Map}(S^4, B\,\mathrm{Sp}(n))$ and $\mathrm{Map}_*(S^4, B\,\mathrm{Sp}(n))$ for all path components (Serre SS of the evaluation fibre sequence collapses).
- Choi–Yoon, 1998: Mod p homology groups of $\mathrm{Map}_*(S^4, B\,\mathrm{F}_4)$.
- Terzić, 2005: rational cohomology of $B\mathcal{G}(P)$ for B a compact simply connected 4-manifold, G a semisimple compact simply connected Lie group.
- Choi, 2006: Mod p homology group of $\mathrm{Map}(S^4, B\,\mathrm{Sp}(n))$ and $\mathrm{Map}_*(S^4, B\,\mathrm{Sp}(n))$ for some path components.

Other cases.

- Kaji, 2006: $H^i(B\mathcal{G}; \mathbb{Z})$ ($i \leq 3$) for G a simply connected compact Lie group and B a closed connected 3-manifold.

7. Other results

- Tsukuda, 1997: Study of maps $BS^1 \to B\mathcal{G}(P)$.
- Tsukuda, 1998: Diffeomorphically isomorphism types of gauge groups.
- Piccinini–Spreafico, 2000: Conjugacy classes in gauge groups.
- Theriault, 2014: Multiplicative decomposition of $\mathcal{G}$ for $G = \mathrm{U}(\infty), \mathrm{SU}(\infty)$ and B a simply connected closed 4-manifold or a closed orientable surface.
- West, 2017: Homotopy types of $B\mathcal{G}_*$, $\mathcal{G}_*$ and $\mathcal{G}$ for gauge transformations over real surfaces.

8. Open problems

On the homotopy types of $\mathcal{G}$.

- Does the finiteness by Crabb–Sutherland (and by Tsutaya) also hold for non-connected compact Lie groups?
  - What can we say about the rationalizations of gauge groups?
- Is there any case when $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_{k'})_{(p)}$ for any prime p but $\mathcal{G}(P_k) \not\simeq \mathcal{G}(P_{k'})$ integrally?
- Classify the homotopy types of $\mathcal{G}(P_k)$ for principal G-bundles over S^{2n_ℓ} where $\{n_1 \leq \cdots \leq n_\ell\}$ is the type of a compact connected Lie group G.
  - In this setting, the homotopy types are completely determined by the p-local homotopy types.
  - For simply connected simple Lie groups, the cases when $G = \mathrm{SU}(n)$ ($n \geq 4$), $\mathrm{Sp}(n)$ ($n \geq 3$), $\mathrm{Spin}(n)$ ($n \geq 7$), G_2, F_4, E_6, E_7, E_8 are still open.

On the homotopy types of $B\mathcal{G}$.

- Does the infiniteness of $B\mathcal{G}$ by Kishimoto–Tsutaya also hold p-locally?

Other problems.

- Find any concrete applications of the study of homotopy types of the gauge groups or their classifying spaces to gauge theory.

9. Unordered list

- Kono, 1991 [Kon91]
  - For $G = \mathrm{SU}(2)$ and $B = S^4$, $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(12, k) = (12, k')$ ($k, k' \in \pi_4(B\,\mathrm{SU}(2)) \cong \mathbb{Z}$).
- Terzic, 2016 [Ter16]
  - The Pontryagin ring of the based loop space of gauge groups and their classifying spaces over $\mathbb{Q}$.
- Cutler, 2018 [Cut18]
  - For $B = S^4$, $G = \mathrm{U}(n), \mathrm{SU}(n)$ ($n \geq 3$) and corresponding principal bundles P_k and P'_k ($k, k' \in \pi_4(B\,\mathrm{U}(n)) \cong \pi_4(B\,\mathrm{SU}(n)) \cong \mathbb{Z}$), $\mathcal{G}(P_k) \cong \mathcal{G}(P'_k) \times S^1$ as topological groups.
  - For $B = S^4$, $G = \mathrm{U}(2), \mathrm{SU}(2)$ and corresponding principal bundles P_k and P'_k ($k, k' \in \pi_4(B\,\mathrm{U}(2)) \cong \pi_4(B\,\mathrm{SU}(2)) \cong \mathbb{Z}$), $\mathcal{G}(P_{2k}) \cong \mathcal{G}(P'_{2k}) \times S^1$ as topological groups.
  - For $B = S^4$, $G = \mathrm{U}(2), \mathrm{PU}(2)$ and corresponding principal bundles P_k and P'_k ($k, k' \in \pi_4(B\,\mathrm{U}(2)) \cong \pi_4(B\,\mathrm{PU}(2)) \cong \mathbb{Z}$), $\mathcal{G}(P_{2k+1}) \cong \mathcal{G}(P'_{2k+1}) \times S^1$ as topological groups.
  - For $G = \mathrm{U}(2)$ and $B = S^4$, $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(12, k) = (12, k')$ ($k, k' \in \pi_4(B\,\mathrm{U}(2)) \cong \mathbb{Z}$).
  - For $G = \mathrm{U}(3)$ and $B = S^4$, $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(24, k) = (24, k')$ ($k, k' \in \pi_4(B\,\mathrm{U}(3)) \cong \mathbb{Z}$).
  - For $G = \mathrm{U}(5)$ and $B = S^4$, $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_{k'})_{(p)}$ for any prime p if and only if $(120, k) = (120, k')$ ($k, k' \in \pi_4(B\,\mathrm{U}(5)) \cong \mathbb{Z}$).
  - For $G = \mathrm{U}(2)$ and $B = \mathbb{C}P^2$, some classification results are given.
- Choi, 2019 [Cho19]
  - For $G = F_4$, $B = S^4$ and $p \neq 2, 13$,
$$H_*(\mathcal{G}(P_k); \mathbb{F}_p) \cong H_*(F_4; \mathbb{F}_p) \otimes H_*(\Omega^4 F_4; \mathbb{F}_p)$$
    as an algebra.
  - For $G = F_4$, $B = S^4$ and $p \neq 2, 13$, the Pontryagin ring $H_*(\mathcal{G}(P_k); \mathbb{F}_p)$ is also determined when $k \not\equiv 0 \bmod p$ and $p = 2, 13$.
- Hasui–Kishimoto–So–Theriault, 2019 [HKST19]
  - For G exceptional p torsion-free except for $(G, p) = (E_7, 5)$ and $B = S^4$, $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_{k'})_{(p)}$ if and only if $\min\{v_p(\gamma(G)), v_p(k)\} = \min\{v_p(\gamma(G)), v_p(k')\}$, where

$$\gamma(G_2) = 3 \cdot 7, \qquad \gamma(F_4) = 5^2 \cdot 13,$$
$$\gamma(E_6) = 5^2 \cdot 7 \cdot 13, \qquad \gamma(E_7) = 7 \cdot 11 \cdot 19, \qquad \gamma(E_8) = 7^2 \cdot 11^2 \cdot 13 \cdot 19 \cdot 31.$$

- Hasui–Kishimoto–Tsutaya, 2019 [HKT19]
  - For G simple, $\mathcal{G}(E_n G)_{(p)}$ is a Sugawara C_k-space if $p > (n + k)n_\ell$.
- Kishimoto–Kono, 2019 [KK19]
  - For $G = \mathrm{Sp}(n)$ and $B = S^4$, $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ implies $(4n(2n+1), k) = (4n(2n+1), k')$ ($k, k' \in \pi_4(B\,\mathrm{Sp}(n)) \cong \mathbb{Z}$).
  - For $G = \mathrm{Sp}(n)$, $B = S^4$ and $(p-1)^2 + 1 \geq 2n$, $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_{k'})_{(p)}$ if and only if $\min\{v_p(4n(2n+1)), v_p(k)\} = \min\{v_p(4n(2n+1)), v_p(k')\}$.
- Membrillo-Solis, 2019 [MS19]
  - For simple G and B a S^3-bundle over S^4, some classification results are given.
  - For $G = \mathrm{SU}(2)$ and $B = S^7$, $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(3, k) = (3, k')$ ($k, k' \in \pi_7(B\,\mathrm{SU}(2)) \cong \mathbb{Z}/12\mathbb{Z}$).

  - For $G = \mathrm{SU}(3)$ and $B = S^7$, $\mathcal{G}(P_k)_{(3)} \simeq \mathcal{G}(P_{k'})_{(3)}$ if and only if $(3, k) = (3, k')$ ($k, k' \in \pi_7(B\,\mathrm{SU}(3)) \cong \mathbb{Z}/6\mathbb{Z}$).
  - For $G = G_2$ and $B = S^7$, $\mathcal{G}(P_k)_{(3)} \simeq \mathcal{G}(P_{k'})_{(3)}$ if and only if $(3, k) = (3, k')$ ($k, k' \in \pi_7(BG_2) \cong \mathbb{Z}/3\mathbb{Z}$).
- Mohammadi–Asadi-Golmankhaneh, 2019 [MAG19]
  - For $G = \mathrm{SU}(4)$ and $B = S^8$, $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ implies $(420, k) = (420, k')$ ($k, k' \in \pi_8(B\,\mathrm{SU}(4)) \cong \mathbb{Z}$); $(3360, k) = (3360, k')$ implies $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_{k'})_{(p)}$.
  - Comment: as the authors pointed out, the classification does not depend on whether M is spin or non-spin.
- So, 2019 [So19]
  - For $G = \mathrm{SU}(n)$ and B a simply-connected closed non-spin 4-manifold, $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ implies $(\frac{1}{2}n(n^2-1), k) = (\frac{1}{2}n(n^2-1), k')$ if n is odd or $(n(n^2-1), k) = (n(n^2-1), k')$ if n is even ($k, k' \in [M, B\,\mathrm{SU}(n)] \cong \pi_4(B\,\mathrm{SU}(n)) \cong \mathbb{Z}$).
  - The order of the Samelson product $S^4 \wedge \mathrm{SU}(n) \to \mathrm{SU}(n)$ is divisible by $\frac{1}{2}n(n^2-1)$ if n is odd or by $n(n^2-1)$ if n is even.
- So–Theriault, 2019 [ST19]
  - For $G = \mathrm{Sp}(2)$ and B a simply-connected closed 4-manifold, $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_{k'})_{(p)}$ for any prime if and only if $(40, k) = (40, k')$ ($k, k' \in [M, B\,\mathrm{Sp}(2)] \cong \pi_4(B\,\mathrm{Sp}(2)) \cong \mathbb{Z}$).
  - Comment: as the authors pointed out, the classification does not depend on whether M is spin or non-spin.
- Theriault, 2019 [The19]
  - For G homotopy commutative and $B = \Sigma X \cup CA$ a homotopy cofiber of sum of Whitehead products between suspension spaces, the following principal fibration splits:

$$\mathrm{Map}_*(\Sigma A, BG) \to \mathrm{Map}_*(B, BG) \to \mathrm{Map}_*(\Sigma X, BG).$$

  - Some applications to gauge groups of this fact are given. Note that $B\mathcal{G}_*(P) \simeq \mathrm{Map}(B, BG)_\alpha$ with the classifying map $\alpha\colon B \to BG$ of P.
- Mohammadi–Asadi-Golmankhaneh, 2020 [MAG20]
  - For $G = \mathrm{SU}(n)$ ($n \geq 3$) and $B = S^6$, $\mathcal{G}(P_{2k}) \simeq \mathcal{G}(P_{2k'})$ implies $((n-1)n(n+1)(n+2), k) = ((n-1)n(n+1)(n+2), k')$ ($k, k' \in \pi_6(B\,\mathrm{SU}(n)) \cong \mathbb{Z}$).
- Huang, 2021 [Hua21]
  - For simple G and B a non-simply-connected closed 5-manifold, some homotopy decompositions are given.
- Kishimoto–Mebrillo-Solis–Theriault, 2021 [KMST21]
  - For $G = (S^3)^n/\{\pm 1\}$ and $B = S^4$, $\mathcal{G}(P_{k_1,\ldots,k_n})_{(p)} \simeq \mathcal{G}(P_{k'_1,\ldots,k'_n})_{(p)}$ for any prime p if and only if $\{(12, k_1), \ldots, (12, k_n)\} = \{(12, k'_1), \ldots, (12, k'_n)\}$ as multisets ($(k_1, \ldots, k_n), (k'_1, \ldots, k'_n) \in \pi_4(B(S^3)^n/\{\pm 1\}) \cong \mathbb{Z}^n$).
  - Comment: Their result is the classification of localized homotopy types. But the method of Kono [Kon91] could improve their result to the one for integral homotopy types.
- Mebrillo-Solis–Theriault, 2021 [MST21]
  - For $G = \mathrm{U}(n)$ ($n < p$) and $B = P^2(p)$, $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ ($k.k' \in [P^2(p), B\,\mathrm{U}(n)] \cong \mathbb{Z}/p\mathbb{Z}$).
  - For $G = \mathrm{U}(p)$ and $B = P^2(p)$, $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(p, k) = (p, k')$ ($k, k' \in [P^2(p), B\,\mathrm{U}(n)] \cong \mathbb{Z}/p\mathbb{Z}$).
  - For $G = \mathrm{U}(p)$ ($p = 3, 5$) and $B = L(p, q)$, $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(p, k) = (p, k')$ ($k, k' \in [L(p, q), B\,\mathrm{U}(n)] \cong \mathbb{Z}/p\mathbb{Z}$).
  - It also contains partial results on other cases.
- Mohammadi, 2021 [Moh21]

  - For $G = \mathrm{PSp}(2)$ and $B = S^4$, $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ implies $(140, k) = (140, k')$ $(k, k' \in \pi_4(B\,\mathrm{Sp}(2)) \cong \mathbb{Z})$; $(140, k) = (140, k')$ implies $\Omega\mathcal{G}(P_k) \simeq \Omega\mathcal{G}(P_{k'})$.
  - For $G = \mathrm{PSp}(3)$ and $B = S^4$, $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ implies $(84, k) = (84, k')$ $(k, k' \in \pi_4(B\,\mathrm{Sp}(3)) \cong \mathbb{Z})$; $(672, k) = (672, k')$ implies $\Omega\mathcal{G}(P_k)_{(p)} \simeq \Omega\mathcal{G}(P_{k'})_{(p)}$ for any prime p.
- Rea, 2021 [Rea21]
  - For $G = \mathrm{PU}(5)$ and $B = S^4$, $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_{k'})_{(p)}$ for any prime p implies $(120, k) = (120, k')$ $(k, k' \in \pi_4(B\,\mathrm{PU}(5)) \cong \mathbb{Z})$.
  - For $G = \mathrm{PU}(3)$ and $B = S^6$, $\mathcal{G}(P_k) \simeq \mathcal{G}(P_{k'})$ if and only if $(120, k) = (120, k')$ $(k, k' \in \pi_6(B\,\mathrm{PU}(3)) \cong \mathbb{Z})$.
  - It also shows the coincidence of the order of the Samelson products $S^{2i} \wedge \mathrm{SU}(n) \to \mathrm{SU}(n)$ and $S^{2i} \wedge \mathrm{PU}(n) \to \mathrm{PU}(n)$.
- Takeda, 2021 [Tak21]
  - For $G = \mathrm{U}(n)$ and $B = S^2$,

$$H^*(B\mathcal{G}(P_k); \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_n, x_1, x_2, \ldots]/(h_n, h_{n+1}, \ldots),$$
$$\text{where } h_i = kc_i + \sum_{j=1}^{i} (-1)^j s_j(x_1, \ldots, x_j) c_{i-j}.$$

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